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FREE PRODUCTS OF TOPOLOGICAL GROUPS WHICH ARE k_ω-SPACES

BY

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ABSTRACT. Let G and H be topological groups and $G * H$ their free product topologized in the manner due to Graev. The topological space $G * H$ is studied, largely by means of its compact subsets. It is established that if G and H are k_ω-spaces (respectively: countable CW-complexes) then so is $G * H$. These results extend to countably infinite free products. If G and H are k_ω-spaces, $G * H$ is neither locally compact nor metrizable, provided G is nondiscrete and H is nontrivial. Incomplete results are obtained about the fundamental group $\pi(G * H)$. If G_1 and H_1 are quotients (continuous open homomorphic images) of G and H, then $G_1 * H_1$ is a quotient of $G * H$.

1. Introduction. In [4], Graev observed that the algebraic free product $G * H$ of two Hausdorff topological groups G and H could be equipped with a topology making it a Hausdorff topological group and satisfying the appropriate conditions for a free product (coproduct) in the category of topological groups; see also [9], [12]. This topology on $G * H$ is the finest topology making $G * H$ a topological group and inducing the original topology on G and H considered as subgroups.

While Hulanicki [5] constructed a compact coproduct by working in the category of compact groups, Ordman [13] proved that the Graev-type free product (the only one we will be concerned with) is never compact. Results there, together with those of Morris [10], [11] made considerable progress in establishing conditions under which free products fail to be locally compact. The first positive results concerning compactness were also contained in [11]: the free product of finitely many locally compact groups is a k-space (for information on k-spaces, see [2], [8], [14]).

Our principal goal is to establish and apply further positive results relating to compactness. We extend the above-mentioned result of Morris to show

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(Theorem 3.2) that a free product of countably many topological groups which are k_ω-spaces is again a k_ω-space (for the definition of k_ω-space, see §3). These seem a very useful class of spaces, since they are also closed under quotients and direct products. We apply this result in §4 by showing that free products of k_ω-spaces are neither locally compact nor metrizable, with best possible connectedness conditions (one factor nondiscrete, one nontrivial).

In §5 we prove (Theorem 5.3) that if each of countably many topological groups is a CW-complex with countably many cells, so is their free product. As an example we discuss $S^1 * S^1$, the free product of two circle groups. In a very natural way, it is a CW-complex with two cells in each nonzero dimension, and fundamental group isomorphic to that of the torus.

In §6 we make the conjecture that if G and H are topological groups, the fundamental groups $\pi(G * H)$ and $(G \times H)$ are equal. We prove in general that the latter is a direct factor of the former, but establish equality only for a limited class of CW-complexes.

Finally, in §7, we establish one more analogy between the free product and direct product by observing that a free product of open continuous homomorphisms is again open.

2. **Notation and preliminaries.** Throughout this paper, the letters G and H will denote Hausdorff topological groups and $G * H$ their topological free product in the sense of [4], [9], [12]. e will be the identity of any group. G' will denote $G\setminus\{e\}$, the set of nonidentity elements of G, and H' will denote $H\setminus\{e\}$. An exponent in an appropriate context will denote direct product; e.g., $(G \times H)^n = (G \times H) \times \cdots \times (G \times H)$.

If $G \times H \times G \times \cdots \times G \times H$ is any finite alternating direct product of G and H (which may begin or end with either) we define $i: G \times H \times \cdots \times G \times H \to G * H$ by

$$i(g_1, h_1, \cdots, g_n, h_n) = g_1 h_1 \cdots g_n h_n.$$

Clearly i is a continuous map of topological spaces (although not a group homomorphism) since multiplication is jointly continuous in $G * H$. Since $i(G)$ is homeomorphic to G and $i(H)$ to H, our notation will confuse them when convenient. We define $\rho_1: G * H \to G$ by $\rho_1(g_1 h_1 \cdots g_n h_n) = g_1 \cdots g_n$; it is a continuous homomorphism. So are the similarly defined maps $\rho_2: G * H \to H$ and $\rho_1 \times \rho_2: G * H \to G \times H$.

Lemma 2.1. *$i: G \times H \to i(G \times H)$ is a homeomorphism.*

Proof. It has inverse, $\rho_1 \times \rho_2$.

So i is a homeomorphism on the sets G, H, $G \times H$, and analogously $H \times G$. i is not a one-to-one map of $G \times H \times G$ into $G * H$, for $i(g_1, e, g_2) = i(g_1 g_2, e, e)$. On the other hand, i is one-to-one when restricted to a set of the general form $(G' \times H')^n$, since i carries each element of such a set to a distinct reduced word (in this particular case, of length $2n$) in $G * H$. It seems reasonable to conjecture that i restricted to such a set is a homeomorphism. We are able to prove this only for a more restricted class of domains.

Lemma 2.2. *Let $A \subset (G' \times H')^n$ and suppose A has compact closure in $(G \times H)^n$. Then $i: A \longrightarrow G * H$ is a homeomorphism onto $i(A)$.*

Proof. By the remarks above, i is continuous and one-to-one. That its inverse is continuous is a variant of the well-known fact about mappings of compact Hausdorff spaces into Hausdorff spaces. Suppose B is a subset of $i(A)$ and $i^{-1}(B) \cap A$ is relatively open in A. Then $i^{-1}(B) = B_0 \cap A$ for some set B_0 open in $\mathrm{cl}(A)$. Since $\mathrm{cl}(A) \setminus B_0$ is then closed and hence compact, its image $i(\mathrm{cl}(A) \setminus B_0)$ is compact and hence closed; thus $i(A) \setminus i(\mathrm{cl}(A) \setminus B_0)$ is open in $i(A)$. We will see $B = i(A) \setminus i(\mathrm{cl}(A) \setminus B_0)$. For, if $b \in B$, $b = i(a)$ uniquely and we need only show $a \notin \mathrm{cl}(A) \setminus B_0$. But $a \in i^{-1}(B) \subset B_0$, completing the argument. Conversely, if $x \in i(A) \setminus i(\mathrm{cl}(A) \setminus B_0)$, $x = i(a)$ for some

$$a \in A \setminus (\mathrm{cl}(A) \setminus B_0) = A \cap B_0 = i^{-1}(B),$$

so $x \in B$. Thus B is equal to the open set $i(A) \setminus i(\mathrm{cl}(A) \setminus B_0)$ whenever $i^{-1}(B) \cap A$ is open, and i is a homeomorphism of A.

It will occasionally be useful to talk of the free product $\Pi^* G_\alpha$ of a (finite or infinite) collection of Hausdorff topological groups G_α for α in some index set A. The following lemma is Theorem 2.5 of [9].

Lemma 2.3. *Let B be a subset of the index set A. The subgroup of $\Pi^* G_\alpha$, $\alpha \in A$, generated by the union of the $i(G_\beta)$, $\beta \in B$, is (1) closed, and (2) homeomorphically isomorphic to $\Pi^* G_\beta$, $\beta \in B$.*

Proof. Let F_1 denote the free product for $\alpha \in A$ and F_2 that for $\beta \in B$. Let $c: F_2 \longrightarrow F_1$ be the inclusion map and define $\sigma: F_1 \longrightarrow F_2$ by extending $\sigma(g) = g$ for $g \in G_\beta$, $\beta \in B$, and $\sigma(g) = e$ for $g \in G_\alpha$, $\alpha \in A \setminus B$. Now $\sigma c: F_2 \longrightarrow F_2$ is the identity map so c is a homeomorphic isomorphism into; $c(F_2)$ is closed in F_1 since it is the kernel of the continuous map $x \longrightarrow \sigma(x)x^{-1}$ from F_1 to F_1.

3. k_ω-spaces. We shall call a topological space X *σ-compact* if $X = \bigcup X_n$ ($n = 1, 2, \cdots$) with each X_n a compact subset of X. Note that X is *not* required to be locally compact. Clearly the X_n may be chosen so that $X_k \subset X_m$ for $k < m$, and will be assumed to be so chosen.

A σ-compact topological space is called a *k_ω-space* with respect to the decomposition $X = \bigcup X_n$, provided than any $A \subset X$ is closed whenever $A \cap X_n$ is compact for every n. The decomposition is essential to the statement, in that some other σ-compact decomposition might not satisfy the condition on closed sets. For information on k_ω-spaces, see [6], [8], [14].

Theorem 3.1. *A finite or countable free product of topological groups is σ-compact if and only if the factors are.*

Proof. *If.* If $G = \bigcup G_n$, $H = \bigcup H_n$ $(n = 1, 2, \cdots)$ display G and H as increasing unions of compact subsets, define $K_n \subset G * H$ to be the subset of $G * H$ consisting of all elements expressible as products of n elements of $G_n \cup H_n$. Then $K_n = i((G_n \cup H_n)^n)$, so K_n is compact; clearly $G * H = \bigcup K_n$ $(n = 1, 2, \cdots)$. In the countable case, if the factors are denoted G_i $(i = 1, 2, \cdots)$, and if $G_i = \bigcup G_{i,n}$ $(n = 1, 2, \cdots)$ is an appropriate decomposition, K_n may be chosen as $i((G_{1,n} \cup \cdots \cup G_{n,n})^n)$.

Only if. If the free product is a k_ω-space, each factor must be a k_ω-space since it is (homeomorphic under i to) a closed subgroup by Lemma 2.3.

To prove that the free product of topological groups which are k_ω-spaces is again a k_ω-space, we must show that if a subset A of (e.g.) $G * H$ has compact intersection with each K_n as defined in the above proof, then A is closed in $G * H$. Morris [11] observed that Theorem 4 of Graev [3] implicitly proves that a free product of locally compact groups is a k-space (and in fact a k_ω-space). We now observe that the same argument works in the present situation. In fact, paralleling Graev, we topologize K_n as a quotient of $(G_n \cup H_n)^n$, and topologize $G * H$ with a topology τ_0 defined by letting U be open in $G * H$ whenever $U \cap K_n$ is relatively open in each K_n. It is easy to check that τ_0 makes $G * H$ a topological space with a topology at least as fine as the free product of topological groups topology τ, and induces the original topology on G and H (i.e., τ_0 restricted to $i(G)$ is homeomorphic to the original topology on G). It remains to check that τ_0 makes $G * H$ a topological group. The confirmation of this is word-for-word as in Graev's proof of Theorem 4 except for the substitution of K_n for F_n and $G * H$ for $F(X)$. Again, there is no extra work in extending to countable products. This shows not only that the free product is a k_ω-space, but that the sets $i((G_n \cup H_n)^n)$ and $i((G_{1,n} \cup G_{2,n} \cup \cdots \cup G_{n,n})^n)$ respectively are the proper compact sets to determine the topology. The theorem we obtain is

Theorem 3.2. *If the topological spaces $G_1, G_2, G_3, \cdots$ are k_ω-spaces, then so is $\Pi^* G_n$ $(n = 1, 2, \cdots)$. If $G_i = \bigcup G_{i,n}$ is an appropriate decomposition of G_i for each i, the product may appropriately be decomposed as $\bigcup (G_{1,n} \cup G_{2,n} \cup \cdots \cup G_{n,n})^n$.*

Conversely, when a free product of topological groups is a k_ω-space, so is each factor; for each factor is a closed subset of the product, and closed subspaces of k_ω-spaces are k_ω-spaces.

Note. The above proof has been substantially improved. Theorem 3.2 is in fact an easy corollary of Theorem 1 of J. Mack, S. A. Morris and E. T. Ordman, *Free topological groups and the projective dimension of a locally compact abelian group* [Proc. Amer. Math. Soc. 40 (1973), 303–308].

3.3. It is natural to ask whether the free product of two topological groups which are k-spaces must be a k-space. To show this is not always true it will suffice to find k-spaces G and H, both topological groups, for which $G \times H$ is not a k-space; then $G * H$ could not be a k-space since $\rho_1 \times \rho_2$: $G * H \longrightarrow G \times H$ is an open map and carries k-spaces to k-spaces. Examples are known of two topological spaces which are k-spaces but whose product is not a k-space; see [1], [2]. The example in [2, pp. 132–133] involves a homogeneous space with an addition but the addition is not jointly continuous (p. 416). In a subsequent paper, it will be shown that if G is the free product of two circle groups and H is the additive group of rationals, G and H are k-spaces but $G \times H$ (and $G * H$) are not k-spaces.

3.4. The difficulty of 3.3 suggests strongly the desirability of carrying out a program like the present one in the category of k-spaces rather than in the category of topological spaces. [Note added: some of this has now been done: see, e.g., E. C. Nummela, *The projective dimension of an abelian K-group*, Proc. Amer. Math. Soc. (to appear).] McCord [7] develops a construction yielding many highly applicable groups whose operations are continuous in the former category. §7 of that paper may be used to construct groups which are k-spaces but whose direct products are not k-spaces; the difficulty lies in determining whether these are topological groups in the traditional sense.

3.5. Every locally compact group is a k-space, since a locally compact space is a k-space.

Every locally compact connected group is a k_ω-space. This follows from [11], but may be proven easily. Let G be locally compact and connected, and let N be a compact neighborhood of e. Then the subgroup of G generated by N is an open and closed subgroup, hence equal to G; hence $G = \bigcup N^n$ $(n = 1, 2, \cdots)$. Thus G is σ-compact. We show that if $A \cap N^n$ is compact for each n, A is closed. Let $x \in \mathrm{cl}(A)$; then for some n, $x \in \mathrm{cl}(A) \cap N^n$. Let $\{x_\delta\}$ be a net in A converging to x; then eventually x_δ is in $xN \subset N^nN = N^{n+1}$. Thus $x \in \mathrm{cl}(A \cap N^{n+1}) = A \cap N^{n+1}$, so $x \in A$.

Further, if G is any locally compact group, G has an open and closed subgroup H which is a k_ω-space. For let N be an open neighborhood of e in G such that $x \in N$ implies $x^{-1} \in N$ and $\mathrm{cl}(N)$ is compact. Then $H = \bigcup N^n = \bigcup \mathrm{cl}(N)^n$ is σ-compact. If $a \in \mathrm{cl}(H) \setminus H$, there is a net $\{x_\delta\}$ in H converging to a. Now eventually some x_δ is in Na. But that x_δ is in some N^n, so $a \in N^{n+1} \subset H$. Hence H is an open and closed subgroup. The confirmation that H is a k_ω-space is exactly as in the above paragraph.

4. Local compactness and metrizability of k_ω-spaces. In [10] and [11], Morris proves that a free product of topological groups fails to be locally compact provided it is an infinite product of groups which are not totally disconnected, or a finite product of connected groups. It is conjectured that a free product of nondiscrete groups is not locally compact (clearly, a free product of discrete groups is discrete and thus locally compact). We are able to eliminate the excess connectivity requirements, at the cost of insisting that two factors be k_ω-spaces.

Lemma 4.1. *Let* $X = \bigcup X_n$ *be a* k_ω*-space, and let* C *be a compact subset of* X*. Then* C *is contained in some* X_n.

Proof. This is a special case of Lemma 9.3 of [14].

Theorem 4.2. *Let* $F = \Pi^* G_\alpha$, $\alpha \in A$, *be any free product of topological groups. Suppose at least one factor is a nondiscrete* k_ω*-space,* G*, and at least one other factor is a nontrivial* k_ω*-space,* H*. Then* F *is not locally compact.*

Proof. If F were locally compact, its closed subgroup (by 2.3) $G * H$ would also be. Let $G = \bigcup G_n$ and $H = \bigcup H_n$ be decompositions of G and H as k_ω-spaces, so that $G * H = \bigcup (G_n \cup H_n)^n$ is a decomposition of $G * H$ as a k_ω-space. Let C be a compact neighborhood of e in $G * H$. By Lemma 4.1, C is contained in some $(G_n \cup H_n)^n$, and thus has no words of length exceeding n. But by Proposition 1 of [12], any neighborhood of e in $G * H$ contains words of arbitrary length. Thus e can have no compact neighborhood.

Under similar hypotheses, we can settle the problem of metrizability.

Lemma 4.3. *If a topological space* X *is a* k_ω*-space, and some point* $x \in X$ *has no compact neighborhood, then* X *is not first countable at* x.

Proof. Let $X = \bigcup X_n$ be an appropriate decomposition of X, and suppose U_n $(n = 1, 2, \cdots)$ is a base for the neighborhood system at x with $U_r \supset U_s$ for $r < s$. No U_r is contained in any X_s, for if it were then $\mathrm{cl}(U_r)$ would be a compact neighborhood of x. For each $n = 1, 2, \cdots$, pick a point $x_n \neq x$ such that $x_n \in U_n \backslash X_n$. The set $\{x_n | n = 1, 2, \cdots\}$ has finite intersection with each X_n, so it is closed in the k_ω-space X; on the other hand it intersects every neighborhood of x. This contradiction shows that no countable base can exist at x.

Theorem 4.4. *Let* F *satisfy the same hypotheses as in Theorem* 4.2. *Then* F *is not metrizable.*

Proof. If F were metrizable, its subgroup $G * H$ would be. But $G * H$ is a k_ω-space by 3.2 and not locally compact by 4.2; hence by 4.3, it is not metrizable.

4.5. *Note.* Subsequently to the initial submission of this paper, Theorems 4.2 and 4.4 were generalized, by S. A. Morris and the author, eliminating the requirement that G and H be k_ω-spaces. This generalization proceeded via a study of commutators: $g\rho([G, H])$, the group generated in $G * H$ by words of the form $g^{-1}h^{-1}gh$.

Two other natural approaches still present themselves. Proposition 1 of [12] asserts that in general, open sets contain very long words. It is possible that compact sets in free products in general, and not just in k_ω-spaces, contain words of bounded length only. A more interesting approach may be motivated by the remarks in 3.5. Suppose G and H are locally compact groups; let G_0 and H_0 respectively be subgroups which are k_ω-spaces and both open and closed. If the natural map $G_0 * H_0 \longrightarrow G * H$ carried $G_0 * H_0$ homeomorphically onto a closed subgroup of $G * H$, then $G_0 * H_0$ would be locally compact as a closed subgroup, but not locally compact by 4.2. Thus, some sort of relative result resembling 7.2 might suffice.

While it is possible that "a free product of subgroups is a subgroup of the free product", a stronger result analogous to the full Kurosh subgroup theorem is not to be expected. In §13 of Graev [3], it is shown that a subgroup of a free topological group is not necessarily again a free topological group.

4.6. Theorem 4.4 guarantees us in particular that the metric topology on the free product $S^1 * S^1$ of two circle groups, considered in [12], [13], is not the free product topology. It also follows that the pseudometric topology constructed by Graev [4] is in general too coarse to be the free product topology.

5. **CW-complexes.** We begin with a discussion of *closure-finite cell complexes.* For definitions and background, see [15]. A closure-finite cell complex is a Hausdorff topological space K which is a union of disjoint subspaces or *cells* each of which is (1) a point, or (2) homeomorphic to the interior of the n-cube $(0, 1)^n$ for some n. Given any n-cell K_i^n, $n \neq 0$, there must be a continuous onto map $[0, 1]^n \longrightarrow \mathrm{cl}(K_i^n)$ which is a homeomorphism from the interior of the cube to K_i^n and for which the image of the boundary of the cube is a finite union of cells of dimension less than n.

Theorem 5.1. *If G and H are closure-finite cell complexes, so is $G * H$.*

Proof. Since every nonempty cell complex has at least one 0-cell, and since G and H are homogeneous, we may decompose them so that $\{e\}$ is a 0-cell in each. We also observe that $G * H$ is equal to either

$$(*) \qquad i(e) \cup i(G) \cup i(H) \cup i(G \times H) \cup i(H \times G) \cup i(G \times H \times G) \cup \cdots$$

$$(**) \qquad i(e) \cup i(G') \cup i(H') \cup i(G' \times H') \cup i(H' \times G') \cup i(G' \times H' \times G') \cup \cdots$$

of the two expressions $(*)$ and $(**)$, where the union is taken over all finite alternating direct products of G and H (respectively G' and H', where $G' = G \setminus \{e\}$). It is routine that each term of $(*)$ is a cell complex; e.g. if $G = \bigcup G_r^n$ and $H = \bigcup H_s^m$, where G_r^n, H_s^m are n- and m-cells respectively, then $G \times H = \bigcup (G_r^n \times H_s^m)$ where $G_r^n \times H_s^m$ is an $(n+m)$-cell. Closure-finiteness is easy to check since

$$\mathrm{cl}(G^n_r \times H^m_s) = \mathrm{cl}(G^n_r) \times \mathrm{cl}(G^m_s).$$

Similar arguments apply to all terms of the union.

Now consider a reduced element $x = g_1 h_1 \cdots g_n h_n \in G * H$. If $g_i \in G^{n_i}_{r_i}$ and $h_j \in H^{m_j}_{s_j}$, then $x \in i(G^{n_1}_{r_1} \times \cdots \times H^{m_k}_{s_k})$, which is an $(n_1 + m_1 + \cdots + n_k + m_k)$-cell. Hence $G * H$ is a union of cells; looking at the expansion in G' and H' shows it is a disjoint union and Lemma 2.2 shows i is a homeomorphism on each open cell. While a given cell in $G * H$ may appear more than once in (*), it is easy to check closure-finiteness from the first appearance in that expression or from (**). □

Actually, we are dealing here with a situation akin to Lemma 7.4 of [7].

A CW-complex is a closure-finite cell complex K topologized so that $A \subset K$ is closed whenever $A \cap \mathrm{cl}(K^n_i)$ is compact for each cell K^n_i. By a countable CW-complex we mean one with countably many cells. If K is a countable CW-complex, $K = K^0_1 \cup K^0_2 \cup \cdots \cup K^1_1 \cup K^1_2 \cup \cdots \cup K^2_1 \cup \cdots$, i.e. there may be countable many cells in each dimension. We may write K as a σ-compact space as follows: $K = \bigcup X_n$ where $X_1 = K^0_1$, $X_2 = \mathrm{cl}(K^1_1) \cup K^0_1 \cup K^0_2$, and in general X_n is the union of the closures of the first s $(n - s)$-cells, for $1 \leq s \leq n$. It is easy to see that the union of the X_n is all of K (since $\mathrm{cl}(K^r_s) \subset X_{r+s}$) and that K with this decomposition is a k_ω-space. This completes half of the following lemma:

Lemma 5.2. *A closure-finite cell complex K is a countable* CW-*complex if and only if it is a k_ω-space with a decomposition $K = \bigcup X_n$ $(n = 1, 2, \cdots)$, each X_n being a finite union of cells of K.*

Proof. "Only if" follows from the above remarks. Conversely, let K be a closure-finite cell complex and a k_ω-space. If an $A \subset K$ has compact intersection with each closed cell of K, it has compact intersection with each X_n and thus is closed in K; hence K is a CW-complex. Since K is a countable union of finite unions of cells, K is a countable complex.

Theorem 5.3. *If G and H are countable* CW-*complexes, so is $G * H$.*

Proof. By Lemma 5.2, G and H are closure-finite cell complexes and k_ω-spaces. By Theorem 5.1, $G * H$ is a closure-finite cell complex. By Theorem 3.2, $G * H$ is a k_ω-space and it is easy to check that the decomposition into compact subsets given there makes the given compact subsets be finite unions of cells. Thus applying Lemma 5.2 again, $G * H$ countable CW-complex.

Clearly, Theorems 5.1 and 5.3 extend without difficulty to countable products.

Example 5.4. We apply the previous theorems to the free product of two circle groups, $S^1 * S^1$. Let G and H denote two copies of the group of additive reals modulo 1, so that $G' = G \setminus \{e\} = (0, 1)$, and consider the expansion (**) from the proof of Theorem 5.1. Each term is mapped homeomorphically into $G * H$, and is an open n-cell $(0, 1)^n$ for appropriate n. The expansion (*) decomposes $G * H$ as a

σ-compact and, in fact, k_ω-space. To complete the description of $G * H$ as a CW-complex, we need only to extend the various maps $i\colon (0, 1)^n \longrightarrow G * H$ to maps $j\colon [0, 1]^n \longrightarrow G * H$, and to see how j maps the boundary of each n-cube. The map j is obvious: $j(r_1, \cdots, r_n) = j(r_1) \cdots j(r_n)$ where $j(0) = j(1) = e$ and, if $r \in (0, 1)$, then $j(r) = i(r)$ where we mean the number $i(r)$ to lie in the same factor G or H as r did.

It is worthwhile to examine the first six terms of (**). The first is $\{e\}$; the second and third attach two 1-cells, leaving a figure eight. We denote the points of this complex by e, $g \in (0, 1) = i(G')$, and $h \in (0, 1) = i(H')$.

The fourth cell corresponds to $G' \times H'$. $j\colon [0, 1]^2 \longrightarrow G * H$ is a homeomorphism on the interior, and maps the boundary of the square by $j(0, h) = h$, $j(g, 1) = g$, $j(1, h) = h$, and $j(g, 0) = g$, where h and g are appropriate points of the figure eight. Clearly these maps agree on the corners (which map to e); hence the boundary of the 2-cell is mapped into the 1-skeleton. At this stage, the CW-complex is a torus.

The fifth cell, corresponding to $H' \times G'$, and its boundary, map into $G * H$ similarly. Again the boundary of the square attaches to the 1-skeleton, and it attaches by a map just like that of the fourth cell except for orientation. Thus at this stage the CW-complex is homeomorphic to two concentric tori with the two equators on one identified with the corresponding equators on the other. Further, the second 2-cell has its boundary mapped onto the same 1-cycle as the first 2-cell; hence it does not affect the fundamental group, which remains $\mathbf{Z} + \mathbf{Z}$. Up to homotopy, the CW-complex is now the wedge (1-point union) of the torus and 2-sphere.

The sixth cell is of form $[0, 1]^3$, corresponding to $G \times H \times G$. Typical of the maps on the six boundary planes are $j(0, h, g) = hg \in i(H \times G)$ and $j(g_1, 0, g_2) = g_1 g_2 \in i(G)$. This term and all following ones involve 3-cells or higher, and thus do not affect the fundamental group, which is $\mathbf{Z} + \mathbf{Z}$.

Hence we see that $S^1 * S^1$ is a CW-complex with one 0-cell and two cells in each higher dimension. After adding the two cells in each finite dimension, the complex is an n-manifold except along the $(n - 1)$-skeleton; once all countably many cells have been added, $S^1 * S^1$ is of course homogeneous. Every open set in $S^1 * S^1$ contains an n-cell for every n.

As a CW-complex, $S^1 * S^1$ is not locally finite. It is not metrizable (and not first countable) although it is clearly separable. We asked in [13]: is $S^1 * S^1$ locally invariant? This question has been answered in the negative by S. A Morris, *Free products of connected locally compact groups are not* SIN *groups* (to appear).

6. **Fundamental groups.** Let G and H continue to be any topological groups. Now $G * H$ is a topological group so its fundamental group $\pi(G * H)$ is abelian. We shall first show that it is at least as big as $\pi(G \times H) = \pi(G) \times \pi(H)$.

Theorem 6.1. $\pi(G * H) = \pi(G \times H) \times L$ *for some group* L *(possibly trivial).*

Proof. Apply the functor π to the sequence $G \times H \longrightarrow G * H \longrightarrow G \times H$ with maps i, $\rho_1 \times \rho_2$, and composite the identity map (see 2.1). This yields maps

$$\pi(G \times H) \longrightarrow \pi(G * H) \longrightarrow \pi(G \times H)$$

with composite the identity map. Since these groups are abelian, $\pi(G \times H)$ is a direct factor of $\pi(G * H)$.

It is reasonable to conjecture that L is always trivial, i.e. that $\pi(G * H) = \pi(G \times H)$. We are able to prove this only for a restricted class of cases; §7 of [7] provides a way of constructing many examples in this class.

Theorem 6.2. *Let* G *and* H *each be a countable* CW*-complex with exactly one* 0*-cell. Then* $\pi(G * H) = \pi(G \times H)$.

Proof. Write $G = \bigcup G_r^n$, $H = \bigcup H_s^m$, as unions of cells, choosing the unique 0-cells G^0 and H^0 to be $\{e\}$. Now $G * H$ is a countable CW-complex with unique 0-cell $\{e\}$, and n-cells arising as finite direct products of cells in G and H. Using the expansion (**) appearing in 5.1, we see that all the 1-cells have been put into $G * H$ once we pass the third term; any cell arising in $G' \times H'$ must be at least a 2-cell. We next add $i(G' \times H')$, which adds 2-cells (and possibly higher dimensional cells) to our 1-complex, introducing relations into its heretofore free fundamental group. In fact, at this point the fundamental group is exactly $\pi(G \times H)$, since by Lemma 2.1, $G \times H$ is homeomorphic to $i(e) \cup i(G') \cup i(H') \cup i(G' \times H') = i(G \times H)$. Now adding $i(H' \times G')$ adds more 2-cells and possibly more relations; following terms add only 3-cells and higher, but do not change the fundamental group. Hence $\pi(G * H)$ is at most some quotient of $\pi(G \times H)$. Since Theorem 6.1 assures us it is no smaller than $\pi(G \times H)$, we conclude $\pi(G * H) = \pi(G \times H)$ as desired.

7. **Quotient maps and coverings.** Since R^1 is the universal covering of S^1, it is tempting to hope that $R^1 * R^1$ will be a covering of $S^1 * S^1$. We shall observe that the most reasonable map from $R^1 * R^1$ to $S^1 * S^1$ is not a local homeomorphism. We shall salvage, and generalize, a weaker result: $S^1 * S^1$ is a quotient of $R^1 * R^1$.

Let $f_i: G_i \longrightarrow H_i$, $i = 1, 2$, be continuous homomorphisms of topological groups. By the free product $f_1 * f_2$ we mean the continuous homomorphism $f_1 * f_2: G_1 * G_2 \longrightarrow H_1 * H_2$ given by, e.g.,

$$f_1 * f_2(g_1h_1 \cdots g_nh_n) = f_1(g_1)f_2(h_1) \cdots f_1(g_n)f_2(h_n).$$

This is simply the unique homomorphism extending f_1 and f_2 to a map defined on $G_1 * G_2$.

Example 7.1. The free product of covering maps need not be a covering map.

Let $f_j: R_j \longrightarrow S_j$ be given by $f_j(t) = \exp(2\pi i t)$ where R_j is a copy of the reals and S_j a copy of the circle, $j = 1, 2$. Now if $x_\delta = (½ + \delta) \in R_1$, $y = 1 \in R_2$, $z_\delta = (½ - \delta) \in R_1$, with $-¼ \leq \delta \leq ¼$, then $x_\delta y z_\delta$ is a reduced word and $\{x_\delta y z_\delta | -¼ \leq \delta \leq ¼\}$ is an arc in $R_1 * R_2$. However,

$$f_1 * f_2(x_\delta y z_\delta) = f_1(x_\delta)f_2(y)f_1(z_\delta) = \exp(2\pi i(½ + \delta))\exp(2\pi i)\exp(2\pi i(½ - \delta))$$
$$= \exp(2\pi i(½ + \delta)) \cdot e \cdot \exp(2\pi i(½ - \delta)) = \exp(2\pi i) = e,$$

the identity of $S_1 * S_2$. Since $\operatorname{Ker}(f_1 * f_2)$ contains an arc, $f_1 * f_2$ is not a local homeomorphism.

We recall a few facts about quotient maps. An onto map $f: X \longrightarrow Y$ of topological spaces is a quotient map provided that $A \subset Y$ is open whenever $f^{-1}(A)$ is open in X. A continuous homomorphism of topological groups is a quotient map if and only if it is an open map, i.e. $f(B)$ is open whenever B is open. A direct product of quotient maps of topological spaces is not necessarily a quotient map [8], but a direct product of quotient (open) continuous homomorphisms of groups is always open and thus quotient. While it is harder to establish, a similar result holds for free products.

Theorem 7.2. *Let $f_i: G_i \longrightarrow H_i$, $i = 1, 2$, be (onto) continuous homomorphisms of topological groups. Then $f_1 * f_2$ is an open map if and only if f_1 and f_2 are open maps.*

Proof. *Only if.* Suppose $f_1 * f_2$ is open. Let $A \subset G_1$ be open; we shall show $f_1(A) \subset H_1$ is open. Define $\nu_1: H_1 * H_2 \longrightarrow H_1$ analogously to $\rho_1: G_1 * G_2 \longrightarrow G_1$. Now ρ_1^{-1}, $f_1 * f_2$, and ν_1 all carry open sets to open sets and thus $\nu_1(f_1 * f_2)\rho_1^{-1}(A)$ is an open subset of H_1. We shall show

$$\nu_1(f_1 * f_2)\rho_1^{-1}(A) = f_1(A).$$

First, let $x \in f_1(A)$ so $x = f_1(a)$; let $\hat{a} \in \rho_1^{-1}(a)$. Then

$$\hat{a} = g_1 g_2 \cdots g_1' g_2' \in G_1 * G_2, \qquad a = \rho_1(\hat{a}) = g_1 \cdots g_1', \qquad x = f_1(a) = f_1(g_1) \cdots f_1(g_1').$$

Also

$$\nu_1(f_1 * f_2)(\hat{a}) = \nu_1(f_1(g_1) \cdots f_2(g_2')) = f_1(g_1) \cdots f_1(g_1') = x.$$

Hence $x \in \nu_1(f_1 * f_2)\rho_1^{-1}(A)$ as desired. On the other hand, if $x \in \nu_1(f_1 * f_2)\rho_1^{-1}(A)$, there is $g_1 g_2 \cdots g_1' g_2' \in G_1 * G_2$ such that

$$a = g_1 \cdots g_1' = \rho_1(g_1 g_2 \cdots g_1' g_2')$$

and

$$x = \nu_1(f_1 * f_2)(g_1 \cdots g_2') = f_1(g_1) \cdots f_1(g_1) = f_1(g_1 \cdots g_1') = f_1(a),$$

so $x \in f_1(A)$ as desired. Thus f_1 is an open map; similarly for f_2.

II. We now suppose f_1 and f_2 are open. $f_1 * f_2$ is a continuous homomorphism. We introduce a new topology τ on $H_1 * H_2$ by letting a subset A of $H_1 * H_2$ be open whenever $(f_1 * f_2)^{-1}(A)$ is open in $G_1 * G_2$. This clearly makes $H_1 * H_2$ a topological space and $f_1 * f_2$ a continuous quotient map of topological spaces. Since $G_1 * G_2$ is a topological group and $H_1 * H_2$ is a topological space and a group (possibly unrelated), the usual argument shows the map $f_1 * f_2$ is an open map (if A is open in $G_1 * G_2$, then $(f_1 * f_2)^{-1}(f_1 * f_2)(A)$ is a union of translates of A, hence open; so $(f_1 * f_2)(A)$ is open). We now check that $H_1 * H_2$ in the new topology τ is a topological group, by checking joint continuity of $(a, b) \to ab^{-1}$. If N is an open set containing ab^{-1}, $(f_1 * f_2)^{-1}(N)$ is open in $G_1 * G_2$. Choose g_1, g_2 preimages of a, b in $G_1 * G_2$; then $g_1 g_2^{-1} \in (f_1 * f_2)^{-1}(N)$. Since $G_1 * G_2$ is a topological group, there are open neighborhoods A of g_1, B of g_2 such that $A \cdot B^{-1} \subset (f_1 * f_2)^{-1}(N)$. Hence $f_1 * f_2(A)$, $f_1 * f_2(B)$ are open neighborhoods of a, b and $(f_1 * f_2(A))(f_1 * f_2(B))^{-1} \subset N$.

Now the free product topology τ_0 on $H_1 * H_2$ is the finest topology on $H_1 * H_2$ making it a topological group and inducing the original topology on each H_i. Thus we can show $f_1 * f_2$ open by showing the topology induced on H_i by τ_0 is at least as fine as that induced by τ, for $i = 1, 2$.

Let N be open in $G_1 * G_2$; then $f_1 * f_2(N)$ is open in τ and $f_1 * f_2(N) \cap H_1$ is open in the topology induced on H_1 by τ. It will suffice to show it is also open in the topology induced on H_1 by τ_0, that is, in the original topology on H_1. Let $h_1 \in f_1 * f_2(N) \cap H_1$. Hence for some $g_1 g_2 \cdots g_1' g_2' \in N$, $h_1 = f_1 * f_2(g_1 \cdots g_2') = f_1(g_1) \cdots f_2(g_2')$. Choose a neighborhood U of $e \in G_1$ such that $Ug_1 g_2 \cdots g_1' g_2' \subset N$. Hence

$$f_1 * f_2(Ug_1 \cdots g_2') = f_1(U) \cdot f_1 * f_2(g_1 \cdots g_2') = f_1(U) \cdot h_1.$$

Now since $f_1 \colon G_1 \to H_1$ is open, $f_1(U)$ is an open neighborhood of $e \in H_1$; say $f_1(U) = V$. We have $f_1(U) \cdot h_1 = V \cdot h_1$, which is a neighborhood of h_1 in H_1. That is, $V \cdot h_1 \subset f_1 * f_2(N) \cap H_1$, and $f_1 * f_2(N) \cap H_1$ is open in the original topology on H_1. The argument for H_2 is similar, completing the argument that $\tau \supset \tau_0$ and $f_1 * f_2$ is open. This completes the proof of 7.2.

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